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# $\pi - \pi$ scattering lengths at finite temperature [*]

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ABSTRACT: The $s$-wave $\pi - \pi$ scattering lengths $a^I(T)$ at finite temperature $T$ and isospin $I = 0, 2$ are calculated within the SU(2) Nambu–Jona-Lasinio model. $a^2(T)$ displays a singularity at the Mott temperature $T_M$, defined as $m_\pi(T_M) = 2m_q(T_M)$, while $a^0(T)$ is singular in addition at the lower temperature $T_d$, where $m_\sigma(T_d) = 2m_\pi(T_d)$, $m_\sigma$ and $m_\pi$ being the masses of the $\sigma$ and $\pi$ mesons, respectively. Numerically we find $T_d = 198\text{MeV}$ and $T_M = 215\text{MeV}$. We speculate on possible experimental consequences.

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The scattering of pion by pions, $\pi + \pi \rightarrow \pi + \pi$ is one of the most fundamental hadronic processes of QCD at a mesonic level. This is so for several reasons. In particular, this scattering involves only the lightest pseudoscalar modes of the theory, that occupy a special role as being the Goldstone particles associated with the almost exact $SU_L(2) \times SU_R(2)$ symmetry that is observed in the particle spectrum. As such, $\pi - \pi$ reactions provide a direct link between the theoretical formalism of chiral symmetry and experiment. This is exemplified in the many existing studies of $\pi - \pi$ scattering using chiral Lagrangians [1-7] that endeavour to calculate scattering amplitudes and scattering lengths. References [1-3] introduce [1] and use [2,3] chiral perturbation theory (ChPT), while [4-6] calculate scattering lengths in the Nambu–Jona-Lasinio (NJL) model [8,9]. Both $s$ and $p$ wave scattering lengths are well reproduced in these models. Essentially, the Weinberg limit [10] can be recovered in all cases, and corrections to it, due to processes such as pion rescattering are accounted for, for example in [2].

To date, investigations performed have all been at zero temperature. Our interest here, however, is to examine $\pi - \pi$ scattering at *finite* temperature, but at zero baryon density, with a view to describing the baryon free central rapidity region occurring in ultra relativistic heavy ion collisions, in which pions are copiously formed. In this calculation, we focus in particular on those temperatures in the vicinity of the phase transition that takes the system from a chirally symmetric state at high temperature to a phase in which chiral symmetry is broken. In our study, we will restrict ourselves to the SU(2) version of the NJL model,

$$\mathcal{L}(x) = \bar{\psi}(x)(i \not{\partial} - m_0)\psi(x) + G[(\bar{\psi}(x)\psi(x))^2 + (\bar{\psi}(x)i\gamma_5\vec{\tau}\psi(x))^2], \qquad (1)$$

where G is a coupling strength of dimension $[\text{Mass}]^{-2}$, and $m_0$ is the current quark mass. Once again, we recall that although this model lacks confinement and is non-renormalizable, it gives a good and transparent description of the low energy meson sector. One simply regulates divergent integrals with a cutoff. In our case, we will do so by restricting the 3-momentum of a quark or antiquark, $|\vec{p}| < \Lambda$. At a finite temperature

$T_c$, this model displays a second order phase transition in the chiral limit ($m_0 = 0$) from a broken to a chirally symmetric state. At finite values of the current quark mass $m_0$, the phase transition is washed out [11]. The temperature range around $T_c$, i.e. $150 - 200$ MeV, is of particular importance for two reasons: (i) current experiments at CERN and AGS energies appear to produce a hadron plasma in this temperature range, and (ii) in the neighbourhood of second order phase transitions, divergences in measurable quantities often occur, such as the phenomenon of critical opalescence [12]. In our case, we might expect that certain cross sections display a rapid temperature dependence. It is therefore of interest to us to study the properties of $\pi - \pi$ scattering in the vicinity of this phase transition. Our calculation of $\pi - \pi$ scattering is performed by keeping only the diagrams from the lowest order expansion in $1/N_c$, with $N_c$ the number of colors [13].

To establish our notation, we briefly summarize the basic formalism of pion-pion scattering. The reader may refer to Refs.[14]-[16] for a more complete discussion. Since three isospin channels are available for the process $\pi\pi \rightarrow \pi\pi$, the invariant scattering amplitude can be written in terms of three unknown functions,

$$< cp_c; dp_d|\mathcal{T}|ap_a; bp_b >= \mathcal{T}_{ab;cd} = A(s,t,u)\delta_{ab}\delta_{cd} + B(s,t,u)\delta_{ac}\delta_{bd} + C(s,t,u)\delta_{ad}\delta_{bc}. \quad (2)$$

Here $a, b$ and $c, d$ are the isospin labels of the initial and final states respectively, and $s, t$ and $u$ are the usual Mandelstam variables, $s = (p_a + p_b)^2$, $t = (p_a - p_c)^2$ and $u = (p_a - p_d)^2$. Perfect crossing symmetry enables one to express $\mathcal{T}_{ab;cd}$ in terms of one amplitude alone, since under the exchange $s \leftrightarrow t$, $A(s,t,u) = B(t,s,u)$, and under $s \leftrightarrow u$, $A(u,t,s) = C(s,t,u)$. It is a standard exercise to project out amplitudes of definite total isospin $I$, which we denote as $\mathcal{T}^I(s,t,u)$. In the limit of scattering at threshold, $\sqrt{s} = 2m_\pi$, $t = u = 0$, the $\mathcal{T}^I$ approach the scattering lengths $a^I$, and one finds the relation

$$a^I = \frac{1}{32\pi}\mathcal{T}^I(s = 4m_\pi^2, t = 0, u = 0). \quad (3)$$

To lowest order in $1/N_c$, the invariant amplitude $\mathcal{T}_{ab;cd}$ is calculated from the box and $\sigma$-propagation diagrams shown in Fig.1. Since these need only be evaluated at threshold, we

3

have $p_i^2 = p^2 = m_\pi^2 = s/4$. This choice of the pion momentum does not restrict generality at $T = 0$ since the system is Lorentz invariant. For $T \neq 0$, the heat bath defines a special system and the specification $p_i^2 = p^2 = s/4$ restricts our results to the scattering of pions whose c.m. system is at rest in the heat bath.

*Box diagram*: The box diagram of Fig.1 is generic for all diagrams which can be constructed with internal quark and antiquark lines. From crossing relations, there are three such possibilities, which we label 1,2 or 3, and which correspond to the diagram shown, and those generated by $s \leftrightarrow t$ and $s \leftrightarrow u$ exchanges, respectively. Here, we follow the notation and calculation of Ref.[6]. A direct evaluation of the diagrams leads one to the result

$$(\mathcal{T}_1)_{ab;cd} = (\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})[4N_cN_f ig_{\pi qq}^4][I(0) + I(p) - p^2 K(p)]$$

$$(\mathcal{T}_2)_{ab;cd} = (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})[4N_cN_f ig_{\pi qq}^4][I(0) + I(p) - p^2 K(p)] \qquad (4)$$

$$(\mathcal{T}_3)_{ab;cd} = (-\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})[8N_cN_f ig_{\pi qq}^4[I(0) + p^4 L(p)/2 - 2p^2 K(p)]$$

in terms of the integrals $I(p)$, $K(p)$ and $L(p)$,

$$I(p) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2][(k+p)^2 - m^2]}$$

$$K(p) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2]^2[(k+p)^2 - m^2]} \qquad (5)$$

$$L(p) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2]^2[(k+p)^2 - m^2]^2}.$$

For $T = 0$, the integrals in Eq.(5) have been evaluated by [6] to obtain the scattering lengths. At finite temperatures, however, one has $\int d^4k/(2\pi)^4 \to \frac{i}{\beta} \sum_n \int d^3k/(2\pi)^3$. Here $\beta$ is the inverse temperature, and the sum on $n$ runs over the Matsubara fermion frequencies $i\omega_n$, $\omega_n = (2n+1)\pi/\beta$, $n = 0, \pm1, \pm2\cdots$, that occur in the 4-vector $k = (i\omega_n, \vec{k})$. Simple analytic expressions can be obtained for these functions, and they are listed in the appendix.

*$\sigma$-propagation diagram:* The diagram with intermediate $\sigma$ propagation shown in Fig. 1 can be simply expressed in terms of its component parts. Again two further diagrams are generated by $s \leftrightarrow t$ and $s \leftrightarrow u$ exchange. Defining the $\sigma - \pi - \pi$ vertex as

$$\Gamma^{\sigma\pi\pi}(p', p) = N_cN_f \int \frac{d^4k}{(2\pi)^4} Tr[S(k+p)\gamma_5 S(k)\gamma_5 S(k+p')], \qquad (6)$$

4

one may easily verify that the required choices of momenta for $s$ and $t$ channel graphs lead to

$$\Gamma^{\sigma\pi\pi}(p, -p) = -8N_c N_f m I(p) \quad (s - \text{channel})$$

$$\text{and} \tag{7}$$

$$\Gamma^{\sigma\pi\pi}(p, p) = -8N_c N_f m [I(0) - p^2 K(p)] \quad (t - \text{channel}).$$

Required also are the meson scattering amplitudes $D_\pi$ and $D_\sigma$, defined as

$$-iD_M(k) = \frac{2iG}{1 - 2G\Pi_M(k)}, \tag{8}$$

for $M = \sigma, \pi$. Since it can be shown that $1 - 2G\Pi_M(k) = m_0/m + 4iGN_c N_f(k^2 - \epsilon_M^2)I(k)$, with $\epsilon_\pi = 0$, $\epsilon_\sigma = 4m^2$, and also that $m_\pi^2 = -m_0/m 4iGN_c N_f I(m_\pi)$, one may express the functions in (8) in terms of $I(p)$ only. One has

$$[-iD_M(k)]^{-1} = 2N_c N_f[(k^2 - \epsilon_M^2)I(k) - m_\pi^2 I(m_\pi)]. \tag{9}$$

Now one may construct the amplitudes for the scattering diagrams. One has

$$(\mathcal{T}_4)_{ab;cd} = \delta_{ab}\delta_{cd} g_{\pi qq}^4 [\Gamma^{\sigma\pi\pi}(p, -p)]^2 D_\sigma(2p) \tag{10}$$

in the $s$-channel, and $s \leftrightarrow t$ and $s \leftrightarrow u$-exchanges give rise to

$$(\mathcal{T}_5)_{ab;cd} = (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) g_{\pi qq}^4 [\Gamma^{\sigma\pi\pi}(p, p)]^2 D_\sigma(0). \tag{11}$$

Finally, expressions (4), (10) and (11) require the temperature dependent pion-quark coupling $g_{\pi qq}$, and which, using the same arguments that led to Eq.(9), is given as $g_{\pi qq}^{-4} = -N^2[I(0) + I(p) - m_\pi^2 K(p)]^2$.

It is now a simple matter to extract the functions $A(s, t, u)$, $B(s, t, u)$ and $C(s, t, u)$ by summing the amplitudes $(\mathcal{T}_i)_{ab;cd}$ of (4), (10) and (11), and identifying $A$,$B$, and $C$ as the coefficients of $\delta_{ab}\delta_{cd}$, $\delta_{ac}\delta_{bd}$ and $\delta_{ad}\delta_{bc}$ respectively. From these, one can construct the $\mathcal{T}^I$, and via Eq.(3), the $s$-wave scattering amplitudes $a^I$ can be obtained. One finds

$$\mathcal{T}^0 = 6\mathcal{T}_1 - \mathcal{T}_3 + 3\mathcal{T}_4 + 2\mathcal{T}_5$$

$$\mathcal{T}^1 = 0 \tag{12}$$

$$\mathcal{T}^2 = 2\mathcal{T}_3 + 2\mathcal{T}_5$$

5

where the $\mathcal{T}_i$ are the functions in Eqs.(4), (10) and (11), here to be understood to be denuded of their isospin cofactors. One observes that $\mathcal{T}^1$ is identically zero, as it must be for s-waves.

Individual contributions to the scattering amplitude are shown as a function of temperature in Fig.2. For this calculation, a standard set of parameters has been used, viz. $\Lambda = 631\text{MeV}$, $m_0 = 5.5\text{MeV}$ and $G = 5.514\text{GeV}^{-2}$ that lead to a value of the current quark mass $m = 339\text{MeV}$, $m_\pi = 139\text{MeV}$ and $f_\pi = 93.3\text{MeV}$. For $m_0 = 0$, these values of $\Lambda$ and $G$ lead to a phase transition at $T_c = 195\text{MeV}$. One observes that the $\mathcal{T}$-matrix amplitudes are constant at low temperatures, and are approximately equal in magnitude at $T = 0$. Structure appears in the amplitudes $\mathcal{T}_i$ at high values of the temperature. $\mathcal{T}_1$ and $\mathcal{T}_3$ are negative and divergent at the Mott temperature $T_M$, defined as the temperature at which the pion mode enters into the continuum, thereby dissociating into constituent quark and antiquark, i.e.

$$m_\pi(T_M) = 2m_q(T_M). \tag{13}$$

The amplitude $\mathcal{T}_5$ is positive and divergent at this temperature. On the other hand $\mathcal{T}_4$ diverges at the temperature $T_d$, at which the $\sigma$-meson can dissociate into pions,

$$m_\sigma(T_d) = 2m_\pi(T_d), \tag{14}$$

due to the s-channel pole that occurs at this point, see Eq.(11). For our set of parameters, we find $T_d = 198\text{MeV}$, while $T_M = 215\text{MeV}$.

In Fig.3, we plot the scattering lengths $a^{I=0,2}$ as a function of the temperature. At $T = 0$, we obtain the numerical values

$$a^0 = 0.179$$

and (15)

$$a^2 = -0.047,$$

in units of the pion mass. These are close to the Weinberg values $(a^0)^W = 7m_\pi^2/(32\pi f_\pi^2) = 0.16$ and $(a^2)^W = -2m_\pi^2/(32\pi f_\pi^2) = 0.044$, as one would expect [4,6], with the difference

being attributable to the composite structure of the mesons. Experimentally, one has [17] $a^0 = 0.26 \pm 0.05$ and $a^2 = -0.028 \pm 0.012$, which are compatible with those from a later reference [18], $a^0 = 0.20 \pm 0.01$ and $a^2 = -0.037 \pm 0.004$. By comparison, chiral perturbation theory for SU(2) [3] gives $a^0 = 0.20$, and $a^2 = -0.042$. It is seen that our results underestimate the ChPT and experimental results. This may be so since our calculation contains only the lowest order terms in $1/N_c$, and does not include any higher order physical processes, such as pion rescattering [2]. We nevertheless have the advantage over ChPT in that we are able to calculate our model results over the entire temperature range.

Regarding the temperature dependence of the $a^I$, we see firstly that $a^0$ varies only slowly with temperature until $T \simeq 140$MeV, and then displays a steep singularity at $T = T_d$, due to the fact that this amplitude contains $\mathcal{T}_4$, or physically, that it contains the $s$-channel pole due to $\sigma$ exchange. It diverges yet again at the Mott temperature $T_M$, since it also contains the amplitudes $T_1$, $T_3$ and $T_5$. On the other hand, $a^2$ is seen to vary slowly with temperature for temperatures less than $T \simeq 200$MeV. Since this scattering length also contains the amplitudes $T_3$ and $T_5$, it exhibits a divergence at the Mott temperature.

We compare our calculation with the zeroth order forms for $(a^I)^W$ given earlier in Eq.(15), by blindly extrapolating them to finite temperature with the replacement $m_\pi \to m_\pi(T)$ and $f_\pi \to f_\pi(T)$, as calculated in the NJL model. A direct calculation of $f_\pi$ at finite temperature [19] indicates that $f_\pi$ is monotonically decreasing with temperature, and diverges at the Mott temperature. Thus one sees that the extrapolated Weinberg estimate follows the calculation for $a^2$ rather closely. On the other hand, it is unable to reproduce the singular structure at $T_d$ for $a^0$. One should also note that this naive extension of the Weinberg formula to finite temperature makes a definite prediction, i.e. that the ratio $a^0(T)/a^2(T) = -3.5$ is a constant, independent of temperature. This appears to hold at the 10% level for temperatures up to $T \simeq 145$ MeV, and is thus an excellent approximation at low temperatures. It breaks down of course in the vicinity of $T_d$ for $a^0$. We comment that

7

similar trends are observed at finite densities, but at $T = 0$ [5], where a phase transition also occurs.

The results shown in Fig.3 have been obtained in a particular model, the NJL model, which has proven rather reliable at the temperature $T = 0$. One may ask the question to what degree the results in the neighbourhood of the phase transition can be trusted, in particular in view of the fact that the NJL model only describes the chiral aspect of the phase transition but not deconfinement. The feature that the scattering lengths diverge at the Mott temperature may have a simple geometrical origin. Physically, when the pion reaches threshold, its constituents become unbound, and the pion radius becomes infinite. This is expressed in the NJL model via the relation of the charge radius to $f_\pi$, i.e. $f_\pi^2 \propto 1/ < r^2 >$, where $< r^2 >$ is the mean pion charge radius squared. Using the Weinberg extrapolation, which according to our calculation is a good (but not perfect) approximation of our calculated results, we have that the $|a(T)|$ increase like $|a(T)| \propto < r^2 > (T)$, with $a = a^0$ or $a^2$. This increase of $|a(T)|$ in the neighbourhood of the chiral transition is thus really a consequence of the pion becoming unbound, i.e. it incorporates implicitly the signal of deconfinement. Therefore the increase of $|a(T)|$, $T \to T_M$ has a rather clearly understood physical origin, and may thus survive any model dependence. However, the *detailed* structure of singularities as shown on Fig.3 may reflect some particular features of the NJL model.

We conclude this paper by speculating as to whether there is a possibility of experimentally verifying the predicted singularities of the $\pi - \pi$ scattering lengths in the neighbourhood of the chiral (and deconfinement) phase transition. The signal observed in the Hanbury-Brown-Twiss (HBT) experiment of $\pi - \pi$ correlations is related to (a) the space-time structure of the pion emitting system, and to (b) the pion-pion final state interaction. Any anomaly observed in the HBT results may be a hint as to the importance of (b), or indirectly the $\pi - \pi$ scattering lengths. One may perhaps even try to compare events with different pion temperatures.

Acknowledgments:

Appendix

The temperature dependent functions $I(p)$, $K(p)$ and $L(p)$ are listed here for the kinematics at threshold, $p = (p_0, \vec{p}) = (\sqrt{s}/2, 0)$. One finds

$$-iI(p_0) = \int \frac{dk}{2\pi^2} \frac{k^2}{E} \frac{f(E) - f(-E)}{p_0^2 - 4E^2}, \qquad (A1)$$

$$-iK(p_0) = -\int \frac{dk}{2\pi^2} \frac{k^2}{4E^3} [f(E) - f(-E)][\frac{1}{p_0^2 - 4E^2} - \frac{8E^2}{(p_0^2 - 4E^2)^2}$$
$$+ 2\beta E f(E) f(-E) \frac{1}{p_0^2 - 4E^2}], \qquad (A2)$$

and

$$-iL(p_0) = -\int \frac{dk k^2}{4\pi^2} \frac{1}{4E^3 p_0^2} [(f(E) - f(-E))[\frac{1}{p_0^2 - 4E^2} - \frac{12E^2}{(p_0^2 - 4E^2)^2} - \frac{64E^4}{(p_0^2 - 4E^2)^3}]$$
$$+ \beta E f(E) f(-E)(\frac{1}{p_0^2 - 4E^2} + \frac{8E^2}{(p_0^2 - 4E^2)^2})] \qquad (A3)$$

In these expressions, $f$ is the Fermi function, $f(E) = [\exp(\beta E) - 1]^{-1}$.

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Figure Captions.

Fig.1: Box and $\sigma$-propagation diagrams for $\pi - \pi$ scattering.

Fig.2: T matrix amplitudes in units of the pion mass, shown as a function of temperature. The Mott temperature $T_M$ and $\sigma$ dissociation temperature $T_d$ are indicated by the dashed vertical lines.

Fig.3: Scattering amplitudes $a^0$ and $a^2$, in units of the pion mass, shown as a function of temperature. The Mott temperature $T_M$ and $\sigma$ dissociation temperature $T_d$ are indicated by the vertical dashed lines. Experimental values (see text) at $T = 0$ are also indicated.

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